

Fixed Points on Tasković's Upper Transversal Intervally Spaces

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ABSTRACT. The main mathematical object of this paper is Tasković's transversal space. There is a new class of spaces among various kinds of spaces with a possibility of big and significant uses. That includes upper transversal intervally spaces as well as the existence of fixed points and the uniqueness with a large number of specific features and can be a model for various uses in various fields of human creation.

1. INTRODUCTION

The main mathematical object of this paper is Tasković's upper transversal space as the furthest space in the hierarchy of geometric spaces. In 1906, in his doctoral thesis, introducing the concept of metric space, Fréchet remarks that instead of triangle inequality a different regularity conditions can also be considered.

Among the various regularity conditions Dj. Kurepa in 1934 replaces triangle inequality with a certain opposite condition defining the corresponding spaces. In 1942 Menger did the same.

Based on these ideas, transversal spaces introduced and defined by Milan Tasković in 1998 have appeared. There are upper and lower transversal spaces and almost all geometric spaces are placed in that scope. Fréchet's, Kurepa's and Menger's spaces specifically fit into that concept of consideration. There is a new class of spaces with big and significant uses. That includes upper and lower transversal intervally spaces.

2. TRANSVERSAL UPPER INTERVALLY SPACES

A fundamental first example of upper transversal spaces for the upper bisection function $g : (R_+^0)^2 \rightarrow R_+^0 := [0, \infty)$ defined by $g(s, t) = s + t$ is a *metric space*.

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In connection with the preceding, the function $\rho : X \times X \rightarrow [a, b] \subset R_+^0$ for $a < b$ is called an **upper (intervally) transverse** on X (or *upper intervally transversal*) iff: $\rho[x, y] = \rho[y, x]$ and if there is an **upper (intervally) bisection function** $g : [a, b] \times [a, b] \rightarrow [a, b]$ such that

$$(A) \quad \rho[x, y] \leq \max \{ \rho[x, z], \rho[z, y], g(\rho[x, z], \rho[z, y]) \}$$

for all $x, y, z \in X$. A **transversal upper intervally space** is a set X together with given upper intervally transverse $\rho : X \times X \rightarrow [a, b] \subset R_+^0$ for $a < b$ on X .

In further, a mapping $M : R \rightarrow [a, b] \subset R_+^0$ for $a < b$ is called an **upper (distribution) function** if it is nonincreasing, left-continuous with $\inf M = a$ and $\sup M = b$. We will denote by \mathcal{D} the set of all upper (distribution) functions.

Distributional upper functions were introduced by M. Tasković in paper [7].

Next two spaces are very interesting examples of transversal upper spaces.

First, an **upper statistical space** is a pair (X, \mathcal{R}) , where X is an abstract set and \mathcal{R} is a mapping of $X \times X$ into the set of all upper (distribution) functions \mathcal{D} . We shall denote the upper (distribution) function $\mathcal{R}(u, v)$ by $M_{u,v}(x)$ or $M_{u,v}$, whence the symbol $M_{u,v}(x)$ will denote the value of $M_{u,v}$ at $x \in R$. The functions $M_{u,v}$ are assumed to satisfy the following conditions: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in R$, and

$$(Eq) \quad M_{u,v}(x) = a \text{ for } x > c \text{ if and only if } u = v,$$

and if $M_{u,r}(x) = a$ and $M_{r,v}(y) = a$ implies $M_{u,v}(x + y) = a$ for all $u, v, r \in X$ and for all $x, y \in R$.

In view the condition $M_{u,v}(c) = b$, which evidently, implies that $M_{u,v}(x) = b$ for every $x \leq c$. Thus condition (Eq) is equivalent to the statement: $u = v$ if and only if $M_{u,v}(x) = A(x)$, where $A(x) = b$ if $x \leq c$ and $A(x) = a$ if $x > c$.

Obviously, every metric space may be regarded as an upper statistical space of a special kind. One has only to set $M_{u,v}(x) = A(x - d(u, v))$ for every pair of points (u, v) in the metric space (X, d) . Also, $M_{u,v}(x)$ may be interpreted as the“measure” that the distance between u and v is less than x .

Second example of transversal upper space, an **upper intervally space** (or Tasković's intervally space from [9]) is a nonempty set X together with the functions $M_{u,v}(x)$ with the following properties: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in R$, (Eq) and if there is a nondecreasing function $f : [a, b] \times [a, b] \rightarrow [a, b]$ with the property $f(t, t) \leq t$ for all $t \in [a, b]$ such that

$$(Nt) \quad M_{u,v}(x + y) \leq f(M_{u,r}(x), M_{r,v}(y))$$

for all $u, v, r \in X$ and for all $x, y \geq c$. (Namely, the function $f : [a, b] \times [a, b] \rightarrow [a, b]$ is **nondecreasing** if $a_i, b_i \in [a, b]$ and $a_i \leq b_i$, ($i = 1, 2$), implies $f(a_1, a_2) \leq f(b_1, b_2)$).

We notice, if we chosen an upper bisection (intervally function $g : [a, b] \times [a, b] \rightarrow [a, b]$ such that $g = f$ (from (Nt)), then we immediate obtain that every upper intervalely space, for $\rho[u, v] = M_{u,v}$, is a transversal upper intervalely space; because in this case from (Nt) the following inequalities hold:

$$\begin{aligned} \rho[u, v] &= M_{u,v}(x) \leq f(M_{u,r}(x-y), M_{r,v}(y)) := \\ &:= g(\rho[u, r], \rho[r, v]) \leq \max \{ \rho[u, r], \rho[r, v], g(\rho[u, r], \rho[r, v]) \}. \end{aligned}$$

On the other hand, if: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in R$, (Eq), and if there is a function $\Psi : [a, b] \times [a, b] \rightarrow [a, b]$ such that

$$M_{u,v}(x) \leq \Psi(M_{u,r}(x), M_{r,v}(x))$$

for all $u, v, r \in X$ and for every $x \geq c$, then it is an example of transversal upper intervalely space also.

A mapping $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **Δ -norm** if it satisfies: $\Delta(a, 1) = a$, $\Delta(0, 0) = 0$, $\Delta(a, b) = \Delta(b, a)$, $\Delta(c, d) \geq \Delta(a, b)$ for $c \geq a, d \geq b$ and $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Let \mathcal{B} denote the set of all Δ -norms, partially ordered by $\Delta_1 \leq \Delta_2$ if and only if $\Delta_1(a, b) \leq \Delta_2(a, b)$ for all $a, b \in [0, 1]$ and $\Delta_1, \Delta_2 \in \mathcal{B}$.

In connection with this, an **upper probabilistic MT-space** is a triplet (X, \mathcal{R}, Δ) , where (X, \mathcal{R}) is an upper statistical space and $f \in \mathcal{B}$ satisfies the preceding triangle inequality (Nt).

A very characteristic example, *for further work*, of the transversal upper intervalely spaces is the following space in the following form.

A **transversal upper intervalely T-space** is a pair (X, ρ) , where X is a transversal upper intervalely space and where the upper (intervally) transverse $\rho[u, v] = M_{u,v}(x)$ satisfying: $M_{u,v} = M_{v,u}$, $M_{u,v}(c) = b$ for some $c \in R$ and (Eq).

These spaces were introduced by M. Tasković in paper [7].

3. MAIN RESULT

Theorem 1. *Let (X, ρ) be a complete transversal upper intervalely T-space, where upper transversal is $\rho[u, v] = M_{u,v}(x)$ and upper bisection function $g : [a, b]^2 \rightarrow [a, b]$ is nondecreasing such that $g(t, t) \leq t$. Then the following condition is sufficient for the existence of a unique fixed point of mapping $T : X \rightarrow X$ in the form of:*

(i) (Case $a > 0$ of $[a, b]$). *There is a nondecreasing function $\varphi : [c, +\infty) \rightarrow [c, +\infty) := I_c$ for some fixed $c \in R$ with the properties that*

$$(As) \lim_{n \rightarrow \infty} \varphi^n(t) = +\infty \quad \text{for every } t > c$$

such that for all $u, v \in X$ and for every $x > c$ inequality is true in the form of

$$(1) \quad M_{Tu, Tv}(x) \leq \max \left\{ M_{u,v}(\varphi(x)), \frac{M_{u,Tu}(\varphi(x)) M_{u,Tv}(\varphi(x))}{M_{u,v}(\varphi(x))}, \frac{M_{u,Tv}(\varphi(x)) M_{v,Tu}(\varphi(x))}{a} \right\}$$

Proof. Proof of existence of a fixed point, let $u \in X$ be arbitrary point and let us define the sequence of iterations $u_n = T^n(u)$ for $n \in N \cup \{0\}$. Let us show that the sequence $\{u_n\}_{n \in N \cup \{0\}}$ is fundamental Cauchy's sequence from X . Then it is true for $t > c$ and $m > n$ ($m, n \in N$) according to (A) that

$$(2) \quad M_{u_n, u_m}(t) \leq \max \{ M_{u_n, u_{n+1}}(t), \dots, M_{u_{m-1}, u_m}(t) \}.$$

Since $Tu_n = u_{n+1}$ and $Tu_{n-1} = u_n$ by the change in (1) we get for every $t > c$ the following estimate

$$\begin{aligned} M_{u_n, u_{n+1}}(t) &= M_{Tu_{n-1}, Tu_n}(t) \leq \\ &\leq \max \left\{ M_{u_{n-1}, u_n}(\varphi(t)), \frac{M_{u_{n-1}, Tu_{n-1}}(\varphi(t)) M_{u_{n-1}, Tu_n}(\varphi(t))}{M_{u_{n-1}, u_n}(\varphi(t))}, \frac{M_{u_{n-1}, Tu_n}(\varphi(t)) M_{u_n, Tu_{n-1}}(\varphi(t))}{a} \right\}, \end{aligned}$$

i.e.,

$$\begin{aligned} M_{u_n, u_{n+1}}(t) &= M_{Tu_{n-1}, u_n}(t) \leq \\ &\leq \max \left\{ M_{u_{n-1}, u_n}(\varphi(t)), \frac{M_{u_{n-1}, u_n}(\varphi(t)) M_{u_0, u_{n+1}}(\varphi(t))}{M_{u_{n-1}, u_n}(\varphi(t))}, \frac{M_{u_{n-1}, u_{n+1}}(\varphi(t)) \overbrace{M_{u_n, u_n}(\varphi(t))}^a}{a} \right\} \end{aligned}$$

Since $M_{u,v}(\varphi(t)) = a$ if and only if $u = v$, then from the preceding facts

$$(3) \quad \begin{aligned} M_{u_n, u_{n+1}}(t) &\leq \max \left\{ M_{u_{n-1}, u_n}(\varphi(t)), M_{u_{n-1}, u_{n+1}}(\varphi(t)), \right. \\ &\quad \left. M_{u_{n-1}, u_{n+1}}(\varphi(t)) \right\} \\ &\leq \max \left\{ M_{u_{n-1}, u_n}(\varphi(t)), M_{u_n, u_{n+1}}(\varphi(t)) \right\} \end{aligned}$$

and therefore

$$(4) \quad M_{u_n, u_{n+1}}(\varphi(t)) \leq \max \left\{ M_{u_{n-1}, u_n}(\varphi^2(t)), M_{u_n, u_{n+1}}(\varphi^2(t)) \right\}.$$

According to the estimates (3) and (4) we inductively get for every $k \in N$ that the following inequality is true

$$(5) \quad M_{u_n, u_{n+1}}(\varphi(t)) \leq \max \left\{ M_{u_{n-1}, u_n}(\varphi(t)), M_{u_n, u_{n+1}}(\varphi^k(t)) \right\}$$

i.e. when $k \rightarrow \infty$, we get $M_{u_n, u_{n+1}}(t) \leq M_{u_{n-1}, u_n}(\varphi(t))$ for every $n \in N$,
i.e. we get

$$(6) \quad M_{u_n, u_{n+1}}(t) \leq M_{u_0, u_1}(\varphi^n(t))$$

for every $n \in N$. Therefore, using this fact and previous inequality (2) we get inequality and equality

$$(7) \quad M_{u_n, u_m}(t) \leq \max \left\{ M_{u_0, u_1}(\varphi^n(t)), \dots, M_{u_0, u_1}(\varphi^{m-1}(t)) \right\} = M_{u_0, u_1}(\varphi^n(t)),$$

therefore, we get that $\{u_n\}_{n \in N}$ is one Cachy's or fundamental sequence from X . It means (because of the space X completeness) that there is point $p \in X$ and so $u_n \rightarrow p$, i.e. $T^n(u) \rightarrow p$.

Since (Eq) and the fact $u \neq v$ according to previous conclusions, we get

$$(8) \quad M_{u,v}(\varphi(t)) < M_{u,v}(t) \text{ for some } t > c.$$

From inequality (1) we get

$$(9) \quad M_{T(u_n), T(p)} \leq \max \left\{ M_{u_n, p}(\varphi(t)), \frac{M_{u_n, u_{n+1}}(\varphi(t)) M_{u_n, T(p)}(\varphi(t))}{M_{u_n, p}(\varphi(t))}, \frac{M_{u_n, T(p)}(\varphi(t)) M_{p, u_{n+1}}(\varphi(t))}{a} \right\}$$

for every $t > c$ and every $n \in N$. Therefore as $T(u_n) = u_{n+1}$ for $n \rightarrow \infty$ according to (9), we get $M_{p, T(p)}(t) \leq M_{p, T(p)}(\varphi(t))$ for every $t > c$. This means according to (8) that $Tp = p$.

Let us prove uniqueness of point $p \in X$. Let $Tp = p \neq q = Tq$ by changing in (1)

$$(10) \quad M_{p,q}(x) \leq \max \left\{ M_{p,q}(\varphi(x)), \frac{M_{p,p}(\varphi(x)) M_{p,q}(\varphi(x))}{M_{p,q}(\varphi(x))}, \frac{M_{p,q}(\varphi(x)), M_{q,p}(\varphi(x))}{a} \right\}.$$

Since $M_{p,q}(\varphi(x)) = a$ is equivalent to $p = q$ we have

$$(11) \quad M_{p,q}(x) \leq \max \left\{ M_{p,q}(\varphi(x)), a, \frac{M_{p,q}^2(\varphi(x))}{a} \right\}.$$

from previous we have by induction $M_{p,q}(x) \leq M_{p,q}(\varphi(x)) \leq M_{p,q}(\varphi^2(x)) \leq \dots \leq M_{p,q}(\varphi^n(x))$ which means that, considering the fact $M_{p,q}(\varphi^n(x)) \rightarrow a$ for $n \rightarrow \infty$, we get by changing in(11)

$$M_{p,q}(x) \leq \max \left\{ a, a, \frac{a^2}{a} \right\} = a$$

i.e. $M_{p,q}(x) = a$ is equivalent to $p = q$ we get a contradiction to the our assumption, which means that fixed point of mapping T is unique. From this, the proof of the theorem is complete. \square

Remarks. We notice that in the case $a = 0$ of $[a, b]$ with the conditions of the previous theorem, the following case will be considered as a sufficiently for the existence of at least one fixed point of the mapping $T : X \rightarrow X$:

There is a nondecreasing function $\varphi : I_c \rightarrow I_c$ for some fixed $c \in R$ with properties (As) such that for all $u, v \in X$ ($u \neq v$) and for every $x > c$ inequality is true in the form of

$$(12) \quad \begin{aligned} M_{Tu, Tv}(x) &\leq \frac{M_{u, Tu}(\varphi(x))M_{v, Tv}(\varphi(x))}{M_{u,v}(\varphi(x))} + \\ &+ r(u, v) \frac{M_{u, Tv}(\varphi(x))M_{v, Tu}(\varphi(x))}{M_{u,v}(\varphi(x))} \end{aligned}$$

where $(u, v) \mapsto r(u, v)$ is an arbitrary nonnegative bounded function. If $r(u, v) \leq 1$, then T has a unique fixed point. Also, without of completeness of space X , T has at least one fixed point: by Tasković [10].

Proof. For proof of fixed point existence, let $u \in X$ be arbitrary point and let us define the sequence of iterations $u_n = T^n(u)$ for $n \in N \cup \{0\}$. Let us show that the sequence $\{u_n\}_{n \in N \cup \{0\}}$ is fundamental Cauchy's sequence from X . Then it is true (2) for $t > c$ and $m > n$ ($m, n \in N$) according to (As).

Since $Tu_n = u_{n+1}$ and $Tu_{n-1} = u_n$ by the change in (12) we get for every $t > c$ the following estimate

$$(13) \quad \begin{aligned} M_{u_n, u_{n+1}}(t) &= M_{Tu_{n-1}, Tu_n}(t) \leq \\ &\leq \frac{M_{u_{n-1}, Tu_{n-1}}(\varphi(t))M_{u_n, Tu_n}(\varphi(t))}{M_{u_{n-1}, u_n}(\varphi(t))} + \\ &+ r(u_{n-1}, u_n) \frac{M_{u_{n-1}, Tu_n}(\varphi(t))M_{u_n, Tu_{n-1}}(\varphi(t))}{M_{u_{n-1}, u_n}(\varphi(t))} \\ M_{u_n, u_{n+1}}(t) &\leq \frac{M_{u_{n-1}, u_n}(\varphi(t))M_{u_n, u_{n+1}}(\varphi(t))}{M_{u_{n-1}, u_n}(\varphi(t))} + \\ &+ r(u_{n-1}, u_n) \frac{M_{u_{n-1}, u_{n+1}}(\varphi(t))M_{u_n, u_n}(\varphi(t))}{M_{u_{n-1}, u_n}(\varphi(t))}. \end{aligned}$$

Since $M_{u,v}(\varphi(t)) = a$ if and only if $u = v$, then

$$(14) \quad M_{u_n, u_{n+1}}(t) \leq M_{u_n, u_{n+1}}(\varphi(t)) + r(u_{n-1}, u_n) \frac{M_{u_{n-1}, u_{n+1}}(\varphi(t)) \cdot a}{M_{u_{n-1}, u_n}(\varphi(t))}$$

for $a = 0$ with the condition that $0 \leq r(u_{n-1}, u_n) \leq M < \infty$ we get

$$(15) \quad M_{u_n, u_{n+1}}(t) \leq M_{u_n, u_{n+1}}(\varphi(t)).$$

For every $n \in N$, i.e., by induction we get

$$M_{u_n, u_{n+1}}(t) \leq M_{u_0, u_1}(\varphi^n(t)).$$

Therefore, applying this fact and earlier inequality (2) and inequality (7) we get that $\{u_n\}_{n \in N}$ is one Cauchy's or fundamental sequence from X . It means (because of the space X completeness) that there is point $p \in X$ and so $u_n \rightarrow p$, i.e., $T^n(u) \rightarrow p$.

Since (Eq) and the fact $u \neq v$ according to previous conclusions, we get

$$(16) \quad M_{u,v}(\varphi(t)) < M_{u,v}(t) \text{ for some } t > c.$$

Therefore, since $T(u_n) = u_{n+1}$ for $n \rightarrow \infty$, condition (15) and (16) mean for every $t > c$ that $Tp = p$.

Let us prove uniqueness.

Let us assume that p is not a fixed point but there is such q that $p \neq q$.

Then (i) $Tp = p \neq q = Tq$ by the change in (12) we get

$$(17) \quad M_{p,q}(x) \leq \frac{M_{p,p}(\varphi(x)) M_{q,q}(\varphi(x))}{M_{p,q}(\varphi(x))} + r(p, q) \frac{M_{p,q}(\varphi(x)), M_{q,p}(\varphi(x))}{M_{p,q}(\varphi(x))}.$$

Since for $r(p, q) \leq 1$ is $M_{p,q}(x) \leq M_{p,q}(\varphi(x))$ by induction we get the following inequalities

$$M_{p,q}(x) \leq M_{p,q}(\varphi(x)) \leq \dots \leq M_{p,q}(\varphi^n(x))$$

and when $n \rightarrow \infty$, then $M_{p,q}(\varphi^n(x)) \rightarrow a$. Since $M_{p,p}(\varphi(x)) = a$ by the change in (17) we get

$$(18) \quad M_{p,q}(x) \leq \frac{a \cdot a}{M_{p,q}(\varphi(x))} + r(p, q) M_{p,q}(\varphi(x)),$$

i.e., $p = q$. This is contradicts. The proof is now complete. \square

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